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Inequalities Expressing Degree of Convergence of Rational Functions*†

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If the poles of a sequence of rational functions (of a complex variable) are known precisely or asymptotically, much can be deduced [6] regarding regions of convergence and degree of convergence, but if those poles are partially or wholly determined by properties of degree of convergence or of best approximation, little is known of the location of the poles or of regions of convergence. The present communication gives some results on this topic, in particular exhibits an example where the poles of the approximating functions are relatively few in number, yet the function approximated has a natural boundary which does not affect the usual degree of convergence.

The theory of approximation by polynomials to an analytic function relates primarily degree of approximating polynomials and degree of convergence on the one hand to regions of analyticity of the approximated function on the other. The corresponding theory for approximation by rational functions has that same objective, as yet unachieved.

To be more explicit about polynomial approximation, we have

THEOREM 1. *Let E be a closed bounded point set of the z-plane whose complement K is connected, and regular in the sense that it admits a Green's function $G(z)$ with pole at infinity. Let C_R denote generically the locus $G(z) = \log R (>0)$ in K, and E_R its interior. Then a necessary and sufficient condition that the function $f(z)$ be single-valued and analytic in E_ρ $(\rho > 1)$ is that there exist polynomials $p_n(z)$ of respective degrees n such that we have for the Tchebycheff (uniform) norm on E*

$$\limsup_{n\to\infty} \|f(z) - p_n(z)\|^{1/n} \leqq 1/\rho. \tag{1}$$

Theorem 1 is due to S. Bernstein in the case that E is a line segment; the curves C_R are ellipses whose foci are the ends of the segment. For other sets E, steps

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toward the theorem were made by Hilbert, Faber, Montel, Fejér, Szegö, Walsh, and Russell.

Approximation by rational functions is much more difficult. A rational function *of type* (n, ν) is one of the form

$$R_{n\nu}(z) \equiv \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_n}{b_0 z^\nu + b_1 z^{\nu-1} + \ldots + b_\nu}, \qquad \Sigma|b_k| \neq 0. \tag{2}$$

If a function $f(z)$ is continuous on a closed bounded set E without isolated points, then there exists a rational function $R_{n\nu}(z)$ of type (n,ν) of best (*Tchebycheff*) approximation to $f(z)$ on E, which need not be unique:

$$\|f(z) - R_{n\nu}(z)\| \leqq \|f(z) - r_{n\nu}(z)\| \tag{3}$$

for every $r_{n\nu}(z)$ of type (n,ν). Thus there exists a table [*6*] of the $R_{n\nu}(z)$ analogous to that of Padé:

$$\begin{array}{l} R_{00}(z), R_{10}(z), R_{20}(z), \ldots, \\ R_{01}(z), R_{11}(z), R_{21}(z), \ldots, \\ R_{02}(z), R_{12}(z), R_{22}(z), \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \tag{4}$$

The following theorem has been recently established [*1*, *2*, *3*].

THEOREM 2. *Let the point set E satisfy the conditions of Theorem* 1, *and let the function* $f(z)$ *be analytic on E, meromorphic with precisely* ν *poles in* E_ρ. *Then there exist rational functions* $R_{n\nu}(z)$ *of respective types* (n,ν) *such that*

$$\limsup_{n\to\infty} \|f(z) - R_{n\nu}(z)\|^{1/n} \leqq 1/\rho. \tag{5}$$

If (5) *is satisfied, then for n sufficiently large* $R_{n\nu}(z)$ *has precisely* ν *finite poles, which approach the respective poles of* $f(z)$ *in* E_ρ.

The existence of the $R_{n\nu}(z)$ in (5) can readily be proved from Theorem 1. Let $f(z)$ be given, and let $q_\nu(z)$ be the polynomial $z^\nu + \ldots$ whose zeros are the poles of $f(z)$ in E_ρ. Then $f(z)\cdot q_\nu(z)$ is analytic interior to E_ρ and by Theorem 1 there exist polynomials $p_n(z)$ of respective degrees n satisfying

$$\limsup_{n\to\infty} \|f(z) q_\nu(z) - p_n(z)\|^{1/n} \leqq 1/\rho. \tag{6}$$

The function $R_{n\nu}(z) \equiv p_n(z)/q_\nu(z)$ is of type (n,ν), $1/q_\nu(z)$ is bounded on E, so (5) follows from (6).

We shall not prove the last sentence of Theorem 2, but mention that the boundedness of the number of (unprescribed) finite poles of the $R_{n\nu}(z)$ enters strongly. When the sequence $R_{n\nu}(z)$ is given, one proves the existence of a subsequence whose poles have no more than ν finite limit points; it can then be shown that only poles of $f(z)$ in E_ρ can be such limit points. When it is not known that $f(z)$ has precisely a finite number ν of poles in some E_ρ, the behavior of sequences of the $R_{n\mu}(z)$, $\mu \neq 0$, from the table [expression (4)] is not known, and in any case seems very complicated. The primary difficulty is that the position of the finite poles of the $R_{n\mu}(z)$ is not known; the poles may conceivably be everywhere dense in the plane or in some subregions exterior to E. O. Perron has exhibited an example showing that, for the second row of the *Padé* table, the poles may be everywhere dense in the plane.

THEOREM 3. *Let E, C_R, and E_R be as in Theorem* 1. *For fixed μ let every subsequence of a sequence of rational functions $R_{n\mu}(z)$ of respective types (n, μ) converge uniformly on E and also on every closed bounded subset of E_ρ containing no limit point of poles of the $R_{n\mu}(z)$ in that subsequence. Then the limit function $f(z)$ is analytic on E, meromorphic with, at most, μ poles in E_ρ.*

For simplicity in exposition we suppose all the poles of the $R_{n\mu}(z)$ simple; the reader will readily make the necessary modifications for the more general case. We can suppose the poles of the $R_{n\mu}(z)$ to be bounded [*4*]. If the sequence $R_{n\mu}(z)$ has any subsequence with $\mu'(\leqq \mu)$ bounded poles in E_ρ and not approaching C_ρ, it has a subsequence having one pole of each term approaching some point α_1 in E_ρ, it has a new subsequence of that subsequence having two poles of each term approaching two distinct points α_1 and α_2 in E_ρ, and so on, to a subsequence having one pole of each term approaching each of μ' points α_j in E_ρ. This last subsequence has a limit function $f(z)$ having no singularity other than a pole in each of the points α_j, for if the pole β_j of $R_{n\mu}(z)$ approaches α_j, the functions $R_{n\mu}(z) \cdot \prod_j (z - \beta_j)$ are analytic and approach $f(z) \cdot \prod_j (z - \alpha_j)$ uniformly in a neighborhood of α_j. The last subsequence is analytic and converges uniformly in the neighborhood of each point of E_ρ other than the α_j, so $f(z)$ is meromorphic with at most μ poles in E_ρ. This concludes the proof. Of course, not every limit point of poles of the $R_{n\mu}(z)$ need be a pole of $f(z)$.

A related theorem is

THEOREM 4. *Let E, C_R, E_R, and $f(z)$ be as in Theorem* 2. *With fixed μ $(\geqq \nu)$ suppose the rational functions $R_{n\mu}(z)$ of respective types (n, μ) satisfy*

$$\limsup_{n \to \infty} \|f(z) - R_{n\mu}(z)\|^{1/n} \leqq 1/\rho. \tag{7}$$

Then the hypothesis of Theorem 3 *is satisfied. In particular,* (7) *is satisfied by the $R_{n\mu}(z)$ of best approximation to $f(z)$ on E as in* (4).

In the proof of the first part of Theorem 4 we need a comparison sequence; we can use the sequence obtained from (6). The entire remainder of the proof can be given by precisely the method of proof of [3], Lemma 1; compare also [4]. The second part of Theorem 4 ([5], Theorem 5) follows from the monotonic character of the norms relating to the extremal functions of (4); we have

$$\|f(z) - R_{n,\nu}(z)\| \geqq \|f(z) - R_{n+j,\nu+j}(z)\|$$

whenever $j \geqq 0$, $k \geqq 0$.

In the light of Theorems 3 and 4, the further question naturally arises as to convergence of sequences of functions $R_{n,m}(z)$ of respective types (n,m) when both m and n are allowed to become infinite. Here the situation can be quite complicated, and the complications are inherent in the problem, not merely a matter of method of study. We shall show that if $f(z)$ is given analytic on E and meromorphic in E_ρ, then for suitably chosen $f_1(z)$ the limit points of the poles of rational functions (or a subsequence) approximating to $f(z)+f_1(z)$ may form a natural boundary of $f(z)+f_1(z)$ in E_ρ. This phenomenon may arise even if those rational functions $R_{nN_n}(z)$ have relatively few poles, namely $N_n \to \infty$ but $N_n/n \to 0$ as $n \to \infty$.

THEOREM 5. *Let E, C_R, E_R, and $f(z)$ be as in Theorem 1. Then there exists a function $f_1(z)$ analytic on E, with a natural boundary interior to E_ρ such that we have*

$$\limsup_{N_n \to \infty} \|f_1(z) - R_{N_n}(z)\|^{1/N_n} = 0, \tag{8}$$

where the $R_{N_n}(z)$ are rational functions of respective degrees N_n, $N_n \to \infty$, $N_n/n \to 0$ as $n \to \infty$. Consequently if $f(z)$ is analytic on E, meromorphic with no more than ν poles in E_ρ then we have

$$\limsup_{n \to \infty} \|[f(z)+f_1(z)] - [R_{n-N_n,\nu}(z) + R_{N_n}(z)]\|^{1/n} \leqq 1/\rho, \tag{9}$$

where the $R_{n-N_n,\nu}(z)$ of respective types $(n-N_n,\nu)$ are suitably chosen.

We choose distinct points α_n in $E_\rho - E$ approaching from the exterior a small circle γ, which together with its interior lies in $E_\rho - E$, so that every point of γ and no other point is a limit point of the α_n. Choose the indices $N_n(\leqq n)$ as $[n^{1/2}]$ (namely the largest integer not greater than $n^{1/2}$), whence $N_n/n \to 0$, $N_n \to \infty$ as $n \to \infty$; choose $r(0<r<1)$, and choose $A_n = r^{n^n} - r^{(n+1)^{n+1}}$.

We now define $f_1(z)$ by the equation

$$f_1(z) \equiv \sum_1^\infty \frac{A_n}{z-\alpha_n}, \tag{10}$$

so that $f_1(z)$ is defined and analytic on E and indeed at every point of the plane except in the α_n and their limit points. The function $f_1(z)$ defined exterior to γ

is meromorphic there, has a pole in each point α_n, and cannot be continued from the exterior of γ into the interior. Such functions were studied by Borel [7] and have since been studied by various authors, recently by Gonçar [8].

We set

$$R_N(z) \equiv \sum_1^N A_n/(z-\alpha_n),$$

and for z on E we have

$$|f_1(z) - R_N(z)| = \left|\sum_{N+1}^{\infty} A_n/(z-\alpha_n)\right| \leqq r^{(N+1)^{N+1}}/\delta, \tag{11}$$

where $\delta(>0)$ is the distance from E to the nearest α_k or limit point. From (11) we now have with $N = N_n$ an inequality stronger than (8):

$$\limsup_{n\to\infty} \|f_1(z) - R_{N_n}(z)\|^{1/n} = 0. \tag{12}$$

By Theorem 2 there exist rational functions $R_{n\nu}(z)$ of respective types (n,ν) so that (5) is satisfied. A change of notation gives

$$\limsup_{n\to\infty} \|f(z) - R_{n-N_n,\nu}(z)\|^{1/(n-N_n)} \leqq 1/\rho,$$

$$\limsup_{n\to\infty} \|f(z) - R_{n-N_n,\nu}(z)\|^{1/n} \leqq 1/\rho. \tag{13}$$

By (12) and (13) we may write (9). The approximating rational functions in (9) are of type $(n, N_n + \nu)$ if n is sufficiently large, and we have $(N_n + \nu)/n \to 0$, $(N_n + \nu) \to \infty$.

Of course (9) holds also for the rational functions of respective types $(n, N_n + \nu)$ of *best approximation* to $f(z) + f_1(z)$ on E.

Inequality (12) holds not merely for the norm on E, but also for the norm on any closed bounded set containing no point α_k. If D is any closed set which lies in the closure of E_σ and contains no pole of $f(z)$, we have [2, 3] as a consequence of Theorem 2 and inequality (5)

$$\limsup_{n\to\infty} [\max|f(z) - R_{n-N_n,\nu}(z)|,\ z \text{ in } D]^{1/n} \leqq \sigma/\rho,$$

so we have by (12)

$$\limsup_{n\to\infty} [\max|[f(z) + f_1(z)] - [R_{N-N_n,\nu}(z) + R_{N_n}(z)]|,\ z \text{ on } D]^{1/n} \leqq \sigma/\rho.$$

We compare Theorem 5 with Theorems 2, 3, and 4. Under the conditions of Theorem 2, if ρ is the largest number such that $f(z)$ is analytic on E, meromorphic with precisely ν poles in E_ρ, inequality (5) becomes [2, 3] an equality. Thus (5) cannot be improved without modifying the restrictions on $f(z)$.

Yet (9) is determined wholly by the singularities of $f(z)$, not in any way by the singularities of $f_1(z)$ if not on E.

We have stated in Theorem 5 that $f_1(z)$ has a natural boundary in E_ρ; this is of course in the sense of Weierstrass, not in the sense of Borel [*7*]. The function $f_1(z)$ interior to γ and the function $f_1(z)$ exterior to γ are continuous on certain lines crossing γ; indeed, the function $f_1(z)$ in (10) is typical of Borel's quasi-analytic extension of functions along line segments as a consequence of the condition $|A_n|^{1/n} \to 0$. Gonçar [*8*] studies the Hausdorf measures of the sets of singularities of such series as (10).

The purpose of Theorem 5 is to indicate the great contrast in results and methods necessary to change from the theory of approximating rational functions having only a finite number of free (i.e., unprescribed) poles to those having an infinite number of such poles, even if the number is of a lower order of infinity than the degree of the rational functions. Other examples can be given (e.g., [*3*]):

THEOREM 6. *Let E, C_R, and E_R be as in Theorem* 1. *Let the function $g(z)$ be analytic on E, meromorphic in E_ρ, $1 < \rho \leqq \infty$. Then for the rational functions* $S_{nM_n}(z)$ of type (n, M_n), *where $M_n \to \infty$ as $n \to \infty$, of best approximation to $g(z)$ on E, we have*

$$\limsup_{n \to \infty} \|g(z) - S_{n_k M}(z)\|^{1/n} \leqq 1_k \rho. \tag{14}$$

Choose a sequence of numbers ρ_k, $1 < \rho_1 < \rho_2 < \ldots \to \rho$ such that no pole of $g(z)$ lies on $C_{\rho k}$; we denote by n_k the number of poles of $g(z)$ in $E_{\rho k}$. By Theorem 2 we have

$$\limsup_{n \to \infty} \|g(z) - S_{nn_k}(z)\|^{1/n} \leqq 1/\rho_k,$$

for the rational functions $S_{nn_k}(z)$ of type (n, n_k) of best approximation. By the monotonic character of the norms involving rational functions of the table (4) we have

$$\|g(z) - S_{nM_n}(z)\| \leqq \|g(z) - S_{nn_k}(z)\|$$

for each fixed n_k and for n sufficiently large. Thus the first member of (14) for every k is not greater than $1/\rho_k$, so (14) follows.

The function $g(z)$ of Theorem 6 can be combined with the function $f_1(z)$ of Theorem 5 to yield a new result:

THEOREM 7. *Let E, C_R, and E_R be as in Theorem* 1. *Then for $f_1(z)$ and $R_{N_n}(z)$ of Theorem* 5 *and $g(z)$ and $S_{nM_n}(z)$ of Theorem* 6 *we have*

$$\limsup_{n \to \infty} \|[f_1(z) + g(z)] - [R_{N_n}(z) + S_{nM_n}(z)]\|^{1/n} \leqq 1/\rho, \tag{15}$$

where $R_{N_n}(z) + S_{nM_n}(z)$ is a rational function of type $(n + N_n, N_n + M_n)$, and where we have chosen M_n so that $M_n = N_n$, whence $(N_n + M_n)/n \to 0$ as $n \to \infty$.

In particular the second member of (15) may be zero.

The asymptotic behavior of the best approximating functions of type (n, N_n) and of their poles is still unknown.

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